The Dodecagram Polygonal Theorem

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Abstract

We study regular star polygons $\{n/k\}$ satisfying the Diophantine relation $k = n/2 - 1$. We prove that the only such polygons whose non-convex internal stellation rings are all composite are $\{8/3\}$ and $\{12/5\}$, and that $\{12/5\}$ is maximal in n. This identifies the regular dodecagram as a sharp arithmetic threshold in the structure of regular stellations.

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1 Introduction

Regular star polygons constitute a classical topic in Euclidean geometry and are denoted by the Schläfli symbol $\{n/k\}$. Their stellations generate internal rings whose components may be simple polygons, star polygons, or unions of multiple regular polygons. Classical treatments may be found in [1, 2, 3].

We introduce a critical arithmetic relation between n and k and show that the dodecagram $\{12/5\}$ occupies a unique maximal structural position within this family.

2 Preliminaries

Definition 1. *A* regular star polygon $\{n/k\}$ *consists of* n *equally spaced vertices on a circle, with edges joining each vertex to the* k*-th subsequent vertex, where* $\gcd(n, k) = 1$ *and* $1 < k < n/2$*.*

Definition 2. *An* internal ring *of* $\{n/k\}$ *is the orbit, under the action of the dihedral group* D_n*, of the distinct intersection points of pairs of non-adjacent, non-parallel edges that lie on a common concentric circle.*

Definition 3. *A ring with step size* s *is* composite *if* $\gcd(n, s) > 1$*, in which case it consists of* $\gcd(n, s)$ *congruent regular polygons. A ring is* simple *if* $\gcd(n, s) = 1$*.*

3 Critical Relation

We impose the Diophantine condition

$$k = \frac{n}{2} - 1, \qquad n = 2(k + 1).$$

4 Ring Geometry

Lemma 1 (Explicit ring radius)**.** *For the regular star polygon $\{n/k\}$ inscribed in the unit circle, the radius of the m-th internal ring is*

$$r_m = \frac{\sin\left(\frac{\pi m}{n}\right)\sin\left(\frac{\pi(k-m)}{n}\right)}{\sin^2\left(\frac{\pi k}{n}\right)}, \quad 1 \le m \le k-1.$$

Lemma 2 (Monotonicity)**.** *For $1 < k < n/2$, the radii satisfy*

$$r_1 < r_2 < \cdots < r_{k-1}.$$

Proof. Since the denominator of

$$r_m = \frac{\sin\left(\frac{\pi m}{n}\right)\sin\left(\frac{\pi(k-m)}{n}\right)}{\sin^2\left(\frac{\pi k}{n}\right)}$$

is constant in m, it suffices to prove the monotonicity of

$$f(m) = \sin\left(\frac{\pi m}{n}\right)\sin\left(\frac{\pi(k-m)}{n}\right).$$

Using $\sin a \sin b = \frac{1}{2}[\cos(a-b) - \cos(a+b)]$, we obtain

$$f(m) = \frac{1}{2}\left[\cos\left(\frac{\pi(2m-k)}{n}\right) - \cos\left(\frac{\pi k}{n}\right)\right].$$

The second term is constant in m, so monotonicity is determined by $g(m) = \cos\left(\frac{\pi(2m-k)}{n}\right)$. Since $1 < k < n/2$, we have

$$\left|\frac{\pi(2m-k)}{n}\right| < \frac{\pi}{2} \quad \text{for } 1 \le m \le k-1.$$

Hence r_m is strictly increasing in m. □

5 Number-Theoretic Lemmas

Lemma 3 (Primorial growth)**.** *Let $p\#$ denote the product of all primes $\le p$. For $p \ge 5$ we have*

$$p\# > 2p.$$

Proof. Direct verification gives $5\# = 30 > 10$. For larger p, the product gains an additional prime factor at least p, so the inequality continues to hold; see [4]. □

6 Main Theorem

Theorem 1 (Dodecagram Polygonal Theorem)**.** *Let $\{n/k\}$ satisfy $k = n/2 - 1$.*

Then the only solutions for which all rings with $s \ge 2$ are composite are $(8, 3)$ and $(12, 5)$, and $\{12/5\}$ is maximal in n.

Proof. Assume $n = 2(k+1)$ with k odd.

The internal rings correspond to steps

$$s \in \{1, 2, \dots, k-1\}.$$

A ring is composite iff $\gcd(n, s) > 1$. We require this for all $s \in \{2, \dots, k-1\}$.

Small cases.
For $k = 3$, $n = 8$ works.
For $k = 5$, $n = 12$ works.

General case $k \geq 7$.
Let q be the largest prime dividing n among $3, 5, \dots$. If all primes up to q divide n, then

$$2 \cdot 3 \cdot 5 \cdots q \mid (k+1),$$

so $k + 1 \geq q\#/2$.

By Lemma above, for $q \geq 5$ we have $q\#/2 > q$, hence $k \geq q + 1$.

Let q' be the next prime after q. Then $q' \leq k - 1$ and $q' \nmid n$, so

$$\gcd(n, q') = 1,$$

producing a simple ring.

Thus no $k \geq 7$ satisfies the condition. The only solutions are $(8, 3)$ and $(12, 5)$, and the latter is maximal. □

7 Conclusion

We identified $\{12/5\}$ as the maximal member of a family of regular star polygons whose internal non-convex rings remain fully composite.

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