The Dodecagram Polygonal Theorem

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Abstract

We characterize a critical family of regular star polygons described by the Schläfli symbol $\{n/k\}$. Under the Diophantine relation $k = n/2 - 1$, we establish The Dodecagram Polygonal Theorem, proving that the regular dodecagram $\{12/5\}$ is the last polygon for which all non-convex internal stellation rings are composite in the sense of $\gcd(n, s) > 1$. This establishes $\{12/5\}$ as a sharp arithmetic and combinatorial threshold in the theory of regular stellations.

1 Introduction

Regular star polygons constitute a classical topic in Euclidean geometry and are denoted by the Schläfli symbol $\{n/k\}$. Their stellations generate internal rings whose components may be simple polygons, star polygons, or unions of multiple regular polygons.

In this paper we introduce a critical arithmetic relation between n and k and show that the dodecagram $\{12/5\}$ occupies a unique structural position: it is the last polygon in this family whose non-convex internal rings are all composite. This property follows from elementary number-theoretic constraints rather than geometric coincidence.

2 Preliminaries

Definition 1. *A* regular star polygon $\{n/k\}$ *consists of* n *equally spaced vertices on a circle, with edges joining each vertex to the* k*-th subsequent vertex, where* $\gcd(n, k) = 1$ *and* $1 < k < n/2$.

Definition 2. *An* internal ring *of* $\{n/k\}$ *is the orbit, under the action of the dihedral group* D_n*, of intersection points of pairs of non-adjacent edges that lie on a common concentric circle. Rings are indexed from the exterior to the interior.*

Definition 3. *A ring with step size* s *is* composite *if* $\gcd(n, s) > 1$*, in which case it consists of* $\gcd(n, s)$ *congruent regular polygons. A ring is* simple *if* $\gcd(n, s) = 1$.

3 Critical Diophantine Relation

We impose the Diophantine condition

$$k = \frac{n}{2} - 1, \qquad n = 2(k + 1). \tag{1}$$

4 Main Result

Theorem 1 (Dodecagram Polygonal Theorem)**.** *Let $\{n/k\}$ be a regular star polygon satisfying $k = n/2 - 1$. Then:*

1. *The polygon has exactly $k-1$ internal rings with step sizes $s = k-1, \dots, 2, 1$.*
2. *The two integer solutions (n,k) for which all rings with $s \geq 2$ are composite are $(n,k) = (8,3)$. and $(n,k) = (12,5)$.*
3. *The only composite ring of $\{8/3\}$ decompose into two squares; the innermost ring is the convex octagon.*
4. *The three composite rings of $\{12/5\}$ decompose respectively into four equilateral triangles, three squares, and two regular hexagons; the innermost ring is the convex dodecagon.*

5 Proofs

Lemma 1. *If $k = n/2 - 1$ and $\gcd(n,k) = 1$, then k is odd.*

Proof. Since $n = 2(k+1)$, we have $\gcd(n,k) = \gcd(2(k+1),k) = \gcd(2,k)$. Hence $\gcd(n,k) = 1$ if and only if k is odd. □

Lemma 2. *A ring of step size s in $\{n/k\}$ is composite if and only if $\gcd(n,s) > 1$.*

Proof. If $d = \gcd(n,s) > 1$, the traversal closes after n/d steps, yielding d congruent regular polygons. If $d = 1$, all vertices are visited before closure, yielding a single polygon. □

Lemma 3 (Ring Structure)**.** *Let $P_i^{(m)}$ denote the intersection of edge i with edge $i+m$ in $\{n/k\}$, for $1 \leq m \leq k-1$ and indices taken mod n. Then:*

(a) The n points $\{P_i^{(m)}\}_{i=0}^{n-1}$ lie equally spaced on a circle of radius r_m, with $r_1 < r_2 < \cdots < r_{k-1}$.

(b) The ring m has step $s = m$: the points $P_i^{(m)}$ and $P_{i+m}^{(m)}$ are connected by a segment of edge $i+m$.

(c) The j-th ring from the exterior has step $s = k-j$. Thus $\{n/k\}$ has exactly $k-1$ internal rings.

Proof. **(a)** The n-fold rotational symmetry R (rotation by $2\pi/n$) satisfies $R(P_i^{(m)}) = P_{i+1}^{(m)}$, so the n points are equally spaced on a circle of radius $r_m = |P_0^{(m)}|$. Parameterizing the edges and solving the resulting 2×2 linear system shows r_m is strictly increasing in m.

(b) Both $P_0^{(m)}$ and $P_m^{(m)}$ lie on edge m: $P_0^{(m)}$ is the intersection of edge 0 *with* edge m, and $P_m^{(m)}$ is the intersection of edge m *with* edge $2m$. These are the only two intersections of edge m with the ring of radius r_m, and the segment of edge m between them is an edge of ring m. Since $P_i^{(m)}$ occupies angular position i (in units of $2\pi/n$), the step from $P_0^{(m)}$ to $P_m^{(m)}$ is $s = m$.

(c) By (a), the outermost ring is $m = k-1$ and the innermost is $m = 1$. The j-th ring from the exterior is $m = k-j$, with step $s = k-j$. □

Lemma 4 (Uniqueness)**.** *Let k be odd and $n = 2(k+1)$. Then $\gcd(n,s) > 1$ for all $s \in \{2, \dots, k-1\}$ if and only if $(n,k) = (12,5)$ or $(n,k) = (8,3)$.*

Proof. The condition requires that every integer in $\{2, \ldots, k-1\}$ shares a prime factor with $n = 2(k+1)$. Equivalently, letting p_0 be the smallest prime not dividing n, we need $p_0 \geq k$.

Case $(n, k) = (8, 3)$**:** The prime factor of $8 = 2^3$ is 2. The unique factor shared with 8 is 2, and $p_0 = 3 = k$. ✓

Case $(n, k) = (12, 5)$**:** The prime factors of $12 = 2^2 \cdot 3$ are $\{2, 3\}$. Each element of $\{2, 3, 4\}$ shares a factor with 12, and $p_0 = 5 = k$. ✓

Case $k \geq 7$**:** We show $p_0 < k$ by exhaustive sub-case analysis.

- If $3 \nmid n$: then $p_0 = 3$. Since $k \geq 7 > 3$, we have $p_0 < k$.
- If $3 \mid n$ but $5 \nmid n$: then $p_0 = 5$. Since $k \geq 7 > 5$, we have $p_0 < k$.
- If $3 \mid n$ and $5 \mid n$ but $7 \nmid n$: then $p_0 = 7$. Since $15 \mid (k+1)$, we get $k \geq 29 > 7 = p_0$.
- In general: suppose the primes $3, 5, \ldots, q$ all divide n, and $q' \nmid n$ where q' is the next prime after q. Then $2 \cdot 3 \cdot 5 \cdots q \mid (k+1)$, so $k \geq 2 \cdot 3 \cdot 5 \cdots q - 1$. For $q \geq 5$, this bound exceeds q' (since $q' > q$ and $2 \cdot 3 \cdot 5 > 5$), giving $p_0 = q' < k$.

In every sub-case, the smallest prime p_0 not dividing n satisfies $p_0 < k$, so $s = p_0 \in \{2, \ldots, k-1\}$ gives $\gcd(n, p_0) = 1$, a simple ring. ✓ □

6 Geometric Structure and Figures

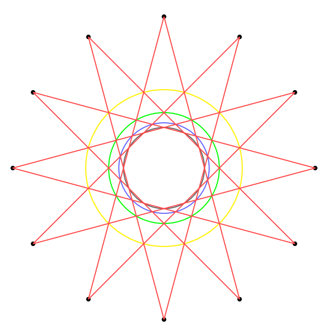

Figure 1: Regular dodecagram $\{12/5\}$ with schematic internal rings.

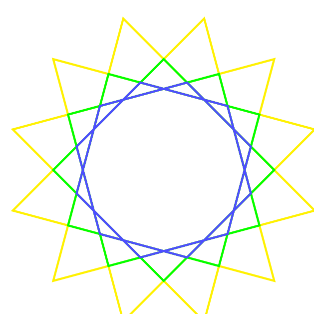

Figure 2: Composite rings of $\{12/5\}$: four equilateral triangles (yellow, r_3), three squares (green, r_2), and two regular hexagons (blue, r_1). Vertex positions match the exact intersection radii and angular offsets.

7 Discussion

The dodecagram appears as a minimal arithmetic configuration in which the prime factors of n cover the full interval of ring step sizes. This property does not persist for larger n, where simple stellations inevitably appear. Thus $\{12/5\}$ constitutes a sharp transition point between fully composite and mixed stellation regimes.

8 Conclusion

We have proven that $\{12/5\}$ is the last regular star polygon satisfying $k = n/2 - 1$ whose non-convex internal rings are all composite. The result follows from elementary number theory and symmetry arguments and suggests further connections between stellation theory and arithmetic geometry.

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